EINSTEIN METRICS ON SPHERES

CHARLES P. BOYER, KRZYSZTOF GALICKI AND JÁNOS KOLLÁR

1. Introduction

Any sphere S^n admits a metric of constant sectional curvature. These canonical metrics are homogeneous and Einstein, that is the Ricci curvature is a constant multiple of the metric. The spheres S^{4m+3}, $m > 1$ are known to have another $Sp(m+1)$-homogeneous Einstein metric discovered by Jensen [Jen73]. In addition, S^{15} has a third Spin(9)-invariant homogeneous Einstein metric discovered by Bourguignon and Karcher [BK78]. In 1982 Ziller proved that these are the only homogeneous Einstein metrics on spheres [Zil82]. No other Einstein metrics on spheres were known until 1998 when Böhm constructed infinite sequences of non-isometric Einstein metrics, of positive scalar curvature, on S^5, S^6, S^7, S^8, and S^9 [Böh98]. Böhm's metrics are of cohomogeneity one and they are not only the first inhomogeneous Einstein metrics on spheres but also the first non-canonical Einstein metrics on even-dimensional spheres. Even with Böhm's result Einstein metrics on spheres appeared to be rare.

The aim of this paper is to demonstrate that on the contrary, at least on odd-dimensional spheres, such metrics occur with abundance in every dimension. Just as in the case of Böhm's construction, ours are only existence results. However, we also answer in the affirmative the long standing open question about the existence of Einstein metrics on exotic spheres. These are differentiable manifolds that are homeomorphic but not diffeomorphic to a standard sphere S^n.

Our method proceeds as follows. For a sequence $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}_+^m$ consider the *Brieskorn–Pham singularity*

$$Y(\mathbf{a}) := \bigl\{ \sum_{i=1}^{m} z_i^{a_i} = 0 \bigr\} \subset \mathbb{C}^m \quad \text{and its link} \quad L(\mathbf{a}) := Y(\mathbf{a}) \cap S^{2m-1}(1).$$

$L(\mathbf{a})$ is a smooth, compact, $(2m-3)$-dimensional manifold. $Y(\mathbf{a})$ has a natural $\mathbb{C}^*$-action and $L(\mathbf{a})$ a natural S^1-action (cf. §33). When the sequence $\mathbf{a}$ satisfies certain numerical conditions, we use the continuity method to produce an orbifold Kähler-Einstein metric on the quotient $(Y(\mathbf{a}) \setminus \{0\})/\mathbb{C}^*$ which then can be lifted to an Einstein metric on the link $L(\mathbf{a})$. We get in fact more:

- The connected component of the isometry group of the metric is S^1.
- We construct continuous families of inequivalent Einstein metrics.
- The Kähler-Einstein structure on the quotient $(Y(\mathbf{a}) \setminus \{0\})/\mathbb{C}^*$ lifts to a *Sasakian-Einstein metric* on $L(\mathbf{a})$. Hence, these metrics have *real Killing spinors* [FK90] which play an important role in the context of p-brane solutions in superstring theory and in M-theory.

In each fixed dimension $(2m-3)$ we obtain a Kähler-Einstein metric on infinitely many different quotients $(Y(\mathbf{a}) \setminus \{0\})/\mathbb{C}^*$, but the link $L(\mathbf{a})$ is a homotopy sphere only for finitely many of them. Both the number of inequivalent families of

Sasakian-Einstein metrics and the dimension of their moduli grow double exponentially with the dimension.

There is nothing special about restricting to spheres or even to Brieskorn-Pham type – our construction is far more general. All the restrictions made in this article are very far from being optimal and we hope that many more cases will be settled in the future. Even with the current weak conditions we get an abundance of new Einstein metrics.

Theorem 1. *On S^5 we obtain 68 inequivalent families of Sasakian-Einstein metrics. Some of these admit non-trivial continuous Sasakian-Einstein deformations.*

The biggest family, constructed in Example 40 has (real) dimension 10.

The metrics we construct are almost always inequivalent not just as Sasakian structures but also as Riemannian metrics. The only exception is that a hypersurface and its conjugate lead to isometric Riemannian metrics, see §20.

In the next odd dimension the situation becomes much more interesting. An easy computer search finds 8,610 distinct families of Sasakian-Einstein structures on standard and exotic 7-spheres. By Kervaire and Milnor there are 28 oriented diffeomorphism types of topological 7-spheres [KM63]. (15 types if we ignore orientation.) The results of Brieskorn allow one to decide which $L(\mathbf{a})$ corresponds to which exotic sphere [Bri66]. We get:

Theorem 2. *All 28 oriented diffeomorphism classes on S^7 admit inequivalent families of Sasakian-Einstein structures, some of them in each diffeomorphism class depending on moduli.*

In each case, the number of families is easily computed and they range from 231 to 452. Moreover, there are fairly large moduli. For example, the standard 7-sphere admits an 82-dimensional family of Sasakian-Einstein metrics, see Example 40. Let us mention here that any orientation reversing diffeomorphism takes a Sasakian-Einstein metric into an Einstein metric, but not necessarily a Sasakian-Einstein metric, since the Sasakian structure fixes the orientation.

Since Milnor's discovery of exotic spheres [Mil56] the study of special Riemannian metrics on them has always attracted a lot of attention. Perhaps the most intriguing question is whether exotic spheres admit metrics of positive sectional curvature. This problem remains open. In 1974 Gromoll and Meyer wrote down a metric of non-negative sectional curvature on one of the Milnor spheres [GM74]. More recently it has been observed by Grove and Ziller that all exotic 7-spheres which are S^3 bundles over S^4 admit cohomogeneity one metrics of non-negative sectional curvature [GZ00]. But it is not known if any of these metrics can be deformed to a metric of strictly positive curvature. Another interesting question concerns existence of metrics of positive Ricci curvature on exotic 7-spheres. This question has now been settled by the result of Wraith who proved that all spheres that are boundaries of parallelizable manifolds admit a metric of positive Ricci curvature [Wra97]. A proof of this result using techniques similar to the present paper was recently given in [BGN03b]. In dimension 7 all homotopy spheres have this property. In this context the result of Theorem 2 can be rephrased as to say that all homotopy 7-spheres admit metrics with positive constant Ricci curvature. Lastly, we should add that although heretofore it was unknown whether Einstein metrics existed on exotic spheres, Einstein metrics have been known to exist on manifolds which are homeomorphic but not diffeomorphic. In dimension 7 there are even examples of homogeneous Einstein metrics with this property [KS88]. Kreck and Stolz find

that there are 7-dimensional manifolds with the maximal number of 28 smooth structures, each of which admits an Einstein metric with positive scalar curvature. Our Theorem 2 establishes the same result for 7-spheres.

In order to organize the higher dimensional cases, note that every link $L(\mathbf{a})$ bounds a parallelizable manifold (called the Milnor fiber). Homotopy n-spheres that bound a parallelizable manifold form a group, called the *Kervaire-Milnor group*, denoted by bP_{n+1}. When $n \equiv 1 \mod 4$ the Kervaire-Milnor group has at most 2 elements, the standard sphere and the *Kervaire sphere*. (It is not completely understood in which dimensions are they different.)

Theorem 3. *For $n \geq 2$, the $(4n+1)$-dimensional standard and Kervaire spheres both admit many families of inequivalent Sasakian-Einstein metrics.*

A partial computer search yielded more than $3 \cdot 10^6$ cases for S^9 and more than 10^9 cases for S^{13}, including a 21300113901610-dimensional family, see Example 45. The only Einstein metric on S^{13} known thus far was the standard one.

In the remaining case of $n \equiv 3 \mod 4$ the situation is more complicated. For these values of n the group bP_{n+1} is quite large (see §29) and we do not know how to show that every member of it admits a Sasakian-Einstein structure, since our methods do not apply to the examples given in [Bri66]. We believe, however, that this is true:

Conjecture 4. *All odd-dimensional homotopy spheres which bound parallelizable manifolds admit Sasakian-Einstein metrics.*

Computer searches are under way to check this in dimensions 11 and 15.

5 (Plan of the proof)**.** Our construction can be divided into four main steps, each of quite different character. The first step, dating back to Kobayashi's circle bundle construction [Kob63], is to observe that a positive Kähler-Einstein metric on the base space of a circle bundle gives an Einstein metric on the total space. This result was generalized to orbifolds giving Sasakian-Einstein metrics in [BG00]. Thus, a positive Kähler-Einstein orbifold metric on $(Y(\mathbf{a}) \setminus \{0\})/\mathbb{C}^*$ yields a Sasakian-Einstein metric on $L(\mathbf{a})$ if $\sum \frac{1}{a_i} > 1$. In contrast with the cases studied in [BG01, BGN03a], our quotients are not well formed, that is, some group elements have codimension 1 fixed point sets.

The second step is to use the continuity method developed by [Aub82, Siu88, Siu87, Tia87] to construct Kähler-Einstein metrics on orbifolds. With minor modifications, the method of [Nad90, DK01] arrives at a sufficient condition, involving the integrability of inverses of polynomials on $Y(\mathbf{a})$.

The third step is to check these conditions. Reworking the earlier estimates given in [JK01, BGN03a] already gives some examples, but here we also give an improvement. This is still, however, quite far from what one would expect.

The final step is to get examples, partly through computer searches, partly through writing down well chosen sequences.

2. Orbifolds as quotients by $\mathbb{C}^*$-actions

Definition 6 (Orbifolds)**.** An *orbifold* is a normal, compact, complex space X locally given by charts written as quotients of smooth coordinate charts. That is, X can be covered by open charts $X = \cup U_i$ and for each U_i there is a smooth complex space V_i and a finite group G_i acting on V_i such that U_i is biholomorphic to the quotient space V_i/G_i. The quotient maps are denoted by $\phi_i : V_i \to U_i$.

The *classical* (or well formed) case is when the fixed point set of every non-identity element of every G_i has codimension at least 2. In this case X alone determines the orbifold structure.

One has to be more careful when there are codimension 1 fixed point sets. (This happens to be the case in all our examples leading to Einstein metrics.) Then the quotient map $\phi_i : V_i \to U_i$ has branch divisors $D_{ij} \subset U_i$ and ramification divisors $R_{ij} \subset V_i$. Let m_{ij} denote the ramification index over D_{ij}. Locally near a general point of R_{ij} the map ϕ_i looks like

$$\mathbb{C}^n \to \mathbb{C}^n, \quad \phi_i : (x_1, x_2, \dots, x_n) \mapsto (z_1 = x_1^{m_{ij}}, z_2 = x_2, \dots, z_n = x_n).$$

Note that

$$\phi_i^*(dz_1 \wedge \cdots \wedge dz_n) = m_{ij} x_1^{m_{ij}-1} \cdot dx_1 \wedge \cdots \wedge dx_n. \tag{6.1}$$

The compatibility condition between the charts that one needs to assume is that there are global divisors $D_j \subset X$ and ramification indices m_j such that $D_{ij} = U_i \cap D_j$ and $m_{ij} = m_j$ (after suitable re-indexing).

It will be convenient to codify these data by a single $\mathbb{Q}$-divisor, called the *branch divisor* of the orbifold,

$$\Delta := \sum (1 - \tfrac{1}{m_j}) D_j.$$

It turns out that the orbifold is uniquely determined by the pair (X, Δ). Slightly inaccurately, we sometimes identify the orbifold with the pair (X, Δ).

In the cases that we consider X is algebraic, the U_i are affine, $V_i \cong \mathbb{C}^n$ and the G_i are cyclic, but these special circumstances are largely unimportant.

Definition 7 (Main examples)**.** Fix (positive) natural numbers $w_1, \dots, w_m$ and consider the $\mathbb{C}^*$-action on $\mathbb{C}^m$ given by $\lambda : (z_1, \dots, z_m) \mapsto (\lambda^{w_1} z_1, \dots, \lambda^{w_m} z_m)$. Set $W = \gcd(w_1, \dots, w_m)$. The Wth roots of unity act trivially on $\mathbb{C}^m$, hence without loss of generality we can replace the action by

$$\lambda : (z_1, \dots, z_m) \mapsto (\lambda^{w_1/W} z_1, \dots, \lambda^{w_m/W} z_m).$$

That is, we can and will assume that the w_i are relatively prime, i.e. $W = 1$. It is convenient to write the m-tuple $(w_1, \cdots, w_m)$ in vector notation as $\mathbf{w} = (w_1, \cdots, w_m)$, and to denote the $\mathbb{C}^*$ action by $\mathbb{C}^*(\mathbf{w})$ when we want to specify the action.

We construct an orbifold by considering the quotient of $\mathbb{C}^m \setminus \{0\}$ by this $\mathbb{C}^*$ action. We write this quotient as $\mathbb{P}(\mathbf{w}) = (\mathbb{C}^m \setminus \{0\})/\mathbb{C}^*(\mathbf{w})$. The orbifold structure is defined as follows. Set $V_i := \{(z_1, \cdots, z_m) \mid z_i = 1\}$. Let $G_i \subset \mathbb{C}^*$ be the subgroup of w_i-th roots of unity. Note that V_i is invariant under the action of G_i. Set $U_i := V_i/G_i$. Note that the $\mathbb{C}^*$-orbits on $(\mathbb{C}^m \setminus \{0\}) \setminus (z_i = 0)$ are in one–to–one correspondence with the points of U_i, thus we indeed have defined charts of an orbifold. As an algebraic variety this gives the *weighted projective space* $\mathbb{P}(\mathbf{w})$ defined as the projective scheme of the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_1, \cdots, z_m]$, where z_i has grading or weight w_i. The weight d piece of $S(\mathbf{w})$, also denoted by $H^0(\mathbb{P}(\mathbf{w}), d)$, is the vector space of weighted homogeneous polynomials of *weighted degree* d. That is, those that satisfy

$$f(\lambda^{w_1} z_1, \dots, \lambda^{w_m} z_m) = \lambda^d f(z_1, \dots, z_m).$$

The weighted degree of f is denoted by $w(f)$.

Let $0 \in Y \subset \mathbb{C}^m$ be a subvariety with an isolated singularity at the origin which is invariant under the given $\mathbb{C}^*$-action. Similarly, we can construct an orbifold on

the quotient $(Y \setminus \{0\})/\mathbb{C}^*(\mathbf{w})$. As a point set, it is the set of orbits of $\mathbb{C}^*(\mathbf{w})$ on $Y \setminus \{0\}$. It's orbifold structure is that induced from the orbifold structure on $\mathbb{P}(\mathbf{w})$ obtained by intersecting the orbifold charts described above with Y. In order to simplify notation, we denote it by $Y/\mathbb{C}^*(\mathbf{w})$ or by $Y/\mathbb{C}^*$ if the weights are clear.

Definition 8. Many definitions concerning orbifolds simplify if we introduce an open set $U_{ns} \subset X$ which is the complement of the singular set of X and of the branch divisor. Thus U_{ns} is smooth and we take $V_{ns} = U_{ns}$.

For the main examples described above U_{ns} is exactly the set of those orbits where the stabilizers are trivial. Every orbit contained in $\mathbb{C}^m \setminus (\prod z_i = 0)$ is such. More generally, a point $(y_1, \dots, y_m)$ corresponds to such an orbit if and only if $\gcd\{w_i : y_i \neq 0\} = 1$.

Definition 9 (Tensors on orbifolds)**.** A *tensor* η on the orbifold (X, Δ) is a tensor η_{ns} on U_{ns} such that for every chart $\phi_i : V_i \to U_i$ the pull back $\phi_i^* \eta_{ns}$ extends to a tensor on V_i. In the *classical case* the complement of U_{ns} has codimension at least 2, so by Hartogs' theorem holomorphic tensors on U_{ns} can be identified with holomorphic tensors on the orbifold. This is not so if there is a branch divisor Δ. We are especially interested in understanding the top dimensional holomorphic forms and their tensor powers.

The *canonical line bundle* of the orbifold $K_{X^{orb}}$ is a family of line bundles one on each chart V_i which is the highest exterior power of the holomorphic cotangent bundle $\Omega^1_{V_i} = T^*_{V_i}$. We would like to study global sections of powers of $K_{X^{orb}}$. Let U_i^{ns} denote the smooth part of U_i and $V_i^{ns} := \phi_i^{-1} U_i^{ns}$. As shown by (6.1), K_{V_i} is not the pull back of K_{U_i}, rather

$$K_{V_i^{ns}} \cong \phi_i^* K_{U_i^{ns}} \left(\sum (m_{ij} - 1) R_{ij}\right).$$

Since $R_{ij} = m_j \phi_i^* D_{ij}$, we obtain, at least formally, that $K_{X^{orb}}$ is the pull back of $K_X + \Delta$, rather than the pull back of K_X. The latter of course makes sense only if we define fractional tensor powers of line bundles. Instead of doing it, we state a consequence of the formula:

Claim 10. For $s > 0$, global sections of $K^{\otimes s}_{X^{orb}}$ are those sections of $K^{\otimes s}_{U_{ns}}$ which have an at most $s(m_i - 1)/m_i$-fold pole along the branch divisor D_i for every i. For $s < 0$, global sections of $K^{\otimes s}_{X^{orb}}$ are those sections of $K^{\otimes s}_{U_{ns}}$ which have an at least $s(m_i - 1)/m_i$-fold zero along the branch divisor D_i for every i.

Definition 11 (Metrics on orbifolds)**.** A *Hermitian metric* h on the orbifold (X, Δ) is a Hermitian metric h_{ns} on U_{ns} such that for every chart $\phi_i : V_i \to U_i$ the pull back $\phi^* h_{ns}$ extends to a Hermitian metric on V_i. One can now talk about curvature, Kähler metrics, Kähler-Einstein metrics on orbifolds.

12 (The hypersurface case)**.** We are especially interested in the case when $Y \subset \mathbb{C}^m$ is a hypersurface. It is then the zero set of a polynomial $F(z_1, \dots, z_m)$ which is equivariant with respect to the $\mathbb{C}^*$-action. F is irreducible since it has an isolated singularity at the origin, and we always assume that F is not one of the z_i. Thus $Y \setminus (\prod z_i = 0)$ is dense in Y.

A differential form on U_{ns} is the same as a $\mathbb{C}^*$-invariant differential form on Y_{ns} and such a form corresponds to a global differential form on X^{orb} iff the corresponding $\mathbb{C}^*$-invariant differential form extends to $Y \setminus \{0\}$.

The $(m-1)$-forms

$$\eta_i := \frac{1}{\partial F/\partial z_i} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_m|_Y$$

satisfy $\eta_i = (-1)^{i-j}\eta_j$ and they glue together to a global generator η of the canonical line bundle $K_{Y\setminus\{0\}}$ of $Y \setminus \{0\}$.

Proposition 13. *Assume that $m \geq 3$ and $s(w(F) - \sum w_i) > 0$. Then the following three spaces are naturally isomorphic:*

(1) *Global sections of $K^{\otimes s}_{X^{orb}}$.*
(2) *$\mathbb{C}^*$-invariant global sections of $K^{\otimes s}_Y$.*
(3) *The space of weighted homogeneous polynomials of weight $s(w(F) - \sum w_i)$, modulo multiples of F.*

Proof. We have already established that global sections of $K^{\otimes s}_{X^{orb}}$ can be identified with $\mathbb{C}^*$-invariant global sections of $K^{\otimes s}_{Y\setminus\{0\}}$. If $m \geq 3$ then Y is a hypersurface of dimension ≥ 2 with an isolated singularity at the origin, thus normal. Hence global sections of $K^{\otimes s}_Y$ agree with global sections of $K^{\otimes s}_{Y\setminus\{0\}}$. This shows the equivalence of (1) and (2).

The $\mathbb{C}^*$-action on η has weight $\sum w_i - w(F)$, thus $K^{\otimes s}_Y$ is the trivial bundle on Y, where the $\mathbb{C}^*$-action has weight $s(\sum w_i - w(F))$. Its invariant global sections are thus given by homogeneous polynomials of weight $s(w(F) - \sum w_i)$ times the generator η. □

In particular, we see that:

Corollary 14. *Notation as in §12. $K^{-1}_{X^{orb}}$ is ample iff $w(F) < \sum w_i$.*

15 (Automorphisms and Deformations)**.** If $m \geq 4$ and $Y \subset \mathbb{C}^m$ is a hypersurface, then by the Grothendieck–Lefschetz theorem, every orbifold line bundle on $Y/\mathbb{C}^*$ is the restriction of an orbifold line bundle on $\mathbb{C}^m/\mathbb{C}^*$ [Gro68]. This implies that every isomorphism between two orbifolds $Y/\mathbb{C}^*(\mathbf{w})$ and $Y'/\mathbb{C}^*(\mathbf{w}')$ is induced by an automorphisms of $\mathbb{C}^m$ which commutes with the $\mathbb{C}^*$-actions. This implies that the weight sequences $\mathbf{w}$ and $\mathbf{w}'$ are the same (up to permutation) and every such automorphism τ has the form

$$\tau(z_i) = g_i(z_1, \ldots, z_m) \quad \text{where} \quad w(g_i) = w_i. \tag{15.1}$$

They form a group $\mathrm{Aut}(\mathbb{C}^m, \mathbf{w})$. For small values of t, maps of the form $\tau(z_i) = z_i + tg_i(z_1, \ldots, z_m)$ where $w(g_i) = w_i$ are automorphisms, hence the dimension of $\mathrm{Aut}(\mathbb{C}^m, \mathbf{w})$ is $\sum_i \dim H^0(\mathbb{P}(\mathbf{w}), w_i)$. Thus we see that, up to isomorphisms, the orbifolds $Y(F)/\mathbb{C}^*$ where $w(F) = d$ form a family of complex dimension at least

$$\dim H^0(\mathbb{P}(\mathbf{w}), d) - \sum_i \dim H^0(\mathbb{P}(\mathbf{w}), w_i), \tag{15.2}$$

and equality holds if the general orbifold in the family has only finitely many automorphisms.

16 (Contact structures)**.** A *holomorphic contact structure* on a complex manifold M of dimension $2n+1$ is a line subbundle $L \subset \Omega^1_M$ such that if θ is a local section of L then $\theta \wedge (d\theta)^n$ is nowhere zero. This forces an isomorphism $L^{n+1} \cong K_M$. We would like to derive necessary conditions for $X^{orb} = Y/\mathbb{C}^*$ to have an orbifold contact structure.

First of all, its dimension has to be odd, so $m = 2n+3$ and $n+1$ must divide the canonical class $K_{X^{orb}} \cong \mathcal{O}(w(F) - \sum w_i)$. If these conditions are satisfied, then a contact structure gives a global section of

$$\Omega^1_{X^{orb}} \otimes \mathcal{O}\left(\tfrac{2}{m-1}(-w(F) + \sum w_i)\right).$$

By pull back, this corresponds to a global section of $\Omega^1_{Y\setminus\{0\}}$ on which $\mathbb{C}^*$ acts with weight $\frac{2}{m-1}(-w(F) + \sum w_i)$.

Next we claim that every global section of $\Omega^1_{Y\setminus\{0\}}$ lifts to a global section of $\Omega^1_{\mathbb{C}^m}$. As a preparatory step, it is easy to compute that $H^i(\mathbb{C}^m \setminus \{0\}, \mathcal{O}_{\mathbb{C}^m\setminus\{0\}}) = 0$ for $0 < i < m-1$. (This is precisely the computation done in [Har77, III.5.1].) Using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{C}^m\setminus\{0\}} \xrightarrow{F} \mathcal{O}_{\mathbb{C}^m\setminus\{0\}} \to \mathcal{O}_{Y\setminus\{0\}} \to 0,$$

these imply that $H^i(Y \setminus \{0\}, \mathcal{O}_{Y\setminus\{0\}}) = 0$ for $0 < i < m-2$. Next apply the $i = 1$ case to the co-normal sequence (cf. [Har77, II.8.12])

$$0 \to \mathcal{O}_{Y\setminus\{0\}} \xrightarrow{dF} \Omega^1_{\mathbb{C}^m\setminus\{0\}}|_{Y\setminus\{0\}} \to \Omega^1_{Y\setminus\{0\}} \to 0$$

to conclude that for $m \geq 4$, every global section of $\Omega^1_{Y\setminus\{0\}}$ lifts to a global section of $\Omega^1_{\mathbb{C}^m\setminus\{0\}}|_{Y\setminus\{0\}}$. The latter is the restriction of the free sheaf $\Omega^1_{\mathbb{C}^m}|_Y$ to $Y \setminus \{0\}$; hence, we can extend the global sections to $\Omega^1_{\mathbb{C}^m}|_Y$ since Y is normal. Finally these lift to global sections of $\Omega^1_{\mathbb{C}^m}$ since $\mathbb{C}^m$ is affine. $\Omega^1_{\mathbb{C}^m} = \sum_i dz_i \mathcal{O}_{\mathbb{C}^m}$, hence there every $\mathbb{C}^*$-eigenvector has weight at least $\min_i\{w_i\}$. So we obtain:

Lemma 17. *The hypersurface $Y/\mathbb{C}^*$ has no holomorphic orbifold contact structure if $m \geq 4$ and $\frac{2}{m-1}(-w(F) + \sum w_i) < \min_i\{w_i\}$.*

This condition is satisfied for all the orbifolds considered in Theorem 34.

3. Sasakian-Einstein structures on links

18 (Brief review of Sasakian geometry)**.** For more details see [BG00] and references therein. Roughly speaking a *Sasakian structure* on a manifold M is a contact metric structure (ξ, η, Φ, g) such that the Reeb vector field ξ is a Killing vector field of unit length, and whose structure transverse to the flow of ξ is Kähler. Here η is a contact 1-form, Φ is a $(1,1)$ tensor field which defines a complex structure on the contact subbundle $\ker \eta$ which annihilates ξ, and the metric is $g = d\eta \circ (\Phi \otimes \mathrm{id}) + \eta \otimes \eta$.

We are interested in the case when both M and the leaves of the foliation generated by ξ are compact. In this case the Sasakian structure is called *quasi-regular*, and the space of leaves X^{orb} is a compact Kähler orbifold [BG00]. M is the total space of a circle *orbi-bundle* (also called V-bundle) over X^{orb}. Moreover, the 2-form $d\eta$ pushes down to a Kähler form ω on X^{orb}. Now ω defines an integral class $[\omega]$ of the orbifold cohomology group $H^2(X^{orb}, \mathbb{Z})$ which generally is only a rational class in the ordinary cohomology $H^2(X, \mathbb{Q})$.

This construction can be inverted in the sense that given a Kähler form ω on a compact complex orbifold X^{orb} which defines an element $[\omega] \in H^2(X^{orb}, \mathbb{Z})$ one can construct a circle orbi-bundle on X^{orb} whose orbifold first Chern class is $[\omega]$. Then the total space M of this orbi-bundle has a natural Sasakian structure (ξ, η, Φ, g), where η is a connection 1-form whose curvature is ω. The tensor field Φ is obtained by lifting the almost complex structure I on X^{orb} to the horizontal distribution

ker η and requiring that Φ annihilates ξ. Furthermore, the map $(M,g)\longrightarrow(X^{orb},h)$ is an orbifold Riemannian submersion.

The Sasakian structure constructed by the inversion process is not unique. One can perform a gauge transformation on the connection 1-form η and obtain a distinct Sasakian structure. However, a straightforward curvature computation shows that there is a unique Sasakian-Einstein metric g with scalar curvature necessarily $2n(2n-1)$ if and only if the Kähler metric h is Kähler-Einstein with scalar curvature $4(n-1)n$, see [Bes87, BG00]. Hence, the correspondence between orbifold Kähler-Einstein metrics on X^{orb} with scalar curvature $4(n-1)n$ and Sasakian-Einstein metrics on M is one-to-one.

19 (Sasakian structures on links of isolated hypersurface singularities)**.** Let F be a weighted homogeneous polynomial as in Definition 7, and consider the subvariety $Y := (F=0)\subset\mathbb{C}^{n+1}$. Suppose further that Y has only an isolated singularity at the origin. Then the link $L_F = F^{-1}(0)\cap S^{2m-1}$ of F is a smooth compact $(m-3)$-connected manifold of dimension $2m-3$ [Mil68]. So if $m\geq 4$ the manifold L_F is simply connected. L_F inherits a circle action from the circle subgroup of the $\mathbb{C}^*$ group described in Definition 7. We denote this circle group by $S^1_{\mathbf{w}}$ to emphasize its dependence on the weights.

As noted in §18 the Kähler structure on $Y/\mathbb{C}^*$ induces a Sasakian structure on the link L_F such that the infinitesimal generator of the weighted circle action defined on $\mathbb{C}^m$ restricts to the Reeb vector field of the Sasakian structure, which we denote by $\xi_{\mathbf{w}}$. This Sasakian structure $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$, which is induced from the *weighted Sasakian structure* on S^{2m-1}, was first noticed by Takahashi [Tak78] for Brieskorn manifolds, and is discussed in detail in [BG01].

The quotient space of the link L_F by this circle action is just the orbifold $X^{orb} = Y/\mathbb{C}^*$ introduced in Definition 7. It has a natural Kähler structure. In fact, all of this fits nicely into a commutative diagram [BG01]:

$$\begin{array}{ccc} L_F & \longrightarrow & S^{2m-1}_{\mathbf{w}} \\ \downarrow{\scriptstyle \pi} & & \\ X^{orb} & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array} \qquad (1)$$

where $S^{2m-1}_{\mathbf{w}}$ emphasizes the weighted Sasakian structure described for example in [BG01], the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. In particular, the Sasakian metric g satisfies $g=\pi^*h+\eta\otimes\eta$, where h is the Kähler metric on X^{orb}.

20 (Isometries of Sasakian structures)**.** Let (X_1^{orb},h_1) and (X_2^{orb},h_2) be two Kähler-Einstein orbifolds and M_1 and M_2 the corresponding Sasakian-Einstein manifolds. As explained in §18, M_1 and M_2 are isomorphic as Sasakian structures iff (X_1^{orb},h_1) and (X_2^{orb},h_2) are biholomorphically isometric. Here we are interested in understanding isometries between M_1 and M_2. As we see, with two classes of exceptions, isometries automatically preserve the Sasakian structure as well.

The exceptional cases are easy to describe:

(1) M_1 and M_2 are both the sphere S^{2n+1} with its round metric. By a theorem of Boothby and Wang, the corresponding circle action is fixed point free [BW58] with weights $(1,\dots,1)$. This happens only in the uninteresting case when $Y\subset\mathbb{C}^m$ is a hyperplane.

(2) M_1 and M_2 have a *3–Sasakian* structure. This means that there is a 2-sphere's worth of Sasakian structures with a transitive action of $SU(2)$ (cf. [BG99] for precise definitions). This happens only if the X_i^{orb} admit holomorphic contact orbifold structures, see [BG97].

Theorem 21. *Let (X_1^{orb}, h_1) and (X_2^{orb}, h_2) be two Kähler-Einstein orbifolds and M_1 and M_2 the corresponding Sasakian-Einstein manifolds. Assume that we are not in either of the exceptional cases enumerated above.*

Let $\phi : M_1 \to M_2$ be an isometry. Then there is an isometry $\bar{\phi} : X_1^{orb} \to X_2^{orb}$ which is either holomorphic or anti-holomorphic, such that the following digram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\phi} & M_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X_1^{orb} & \xrightarrow{\bar{\phi}} & X_2^{orb} \end{array}$$

Moreover, $\bar{\phi}$ determines ϕ up to the S^1-action given by the Reeb vector field.

Proof. Let $\mathcal{S}_i$ denote the Sasakian structure on M_i. Then $\mathcal{S}_1$ and $\phi^*\mathcal{S}_2$ are Sasakian structures on M_1 sharing the same Riemannian metric. Since neither g_1 nor g_2 are of constant curvature nor part of a 3-Sasakian structure, the proof of Proposition 8.4 of [BGN03a] implies that either $\phi^*\mathcal{S}_2 = \mathcal{S}_1$ or $\phi^*\mathcal{S}_2 = \mathcal{S}_1^c$ the conjugate Sasakian structure, $\mathcal{S}_1^c := (-\xi_1, -\eta_1, -\Phi_1, g_1)$. Thus, ϕ intertwines the foliations and gives rise to an orbifold map $\bar{\phi} : X_1^{orb} \longrightarrow X_2^{orb}$ as required.

Conversely, any such biholomorphism or anti-biholomorphism $\bar{\phi}$ lifts to an orbi-bundle map $\phi : M_1 \longrightarrow M_2$ uniquely up to the S^1-action given by the Reeb vector field. □

Putting this together with §15 we obtain:

Corollary 22. *Let $Y_1 \subset \mathbb{C}^m$ (resp. $Y_2 \subset \mathbb{C}^m$) be weighted homogeneous hypersurfaces with isolated singularities at the origin with weights $\mathbf{w}_1$ (resp. $\mathbf{w}_2$). Assume that*

(1) *$m \geq 4$.*
(2) *Y_1, Y_2 have isolated singularities at the origin.*
(3) *$Y_1/\mathbb{C}^*(\mathbf{w}_1)$ and $Y_2/\mathbb{C}^*(\mathbf{w}_2)$ both have Kähler-Einstein metrics.*
(4) *Neither $Y_1/\mathbb{C}^*(\mathbf{w}_1)$ nor $Y_2/\mathbb{C}^*(\mathbf{w}_2)$ has a holomorphic contact structure.*

Let (L_1, g_1) and (L_2, g_2) be the corresponding Einstein metrics on the links. Then

(5) *The connected component of the isometry group of (L_i, g_i) is the circle S^1.*
(6) *(L_1, g_1) and (L_2, g_2) are isometric iff $\mathbf{w}_1 = \mathbf{w}_2$ (up to permutation) and there is an automorphism $\tau \in \mathrm{Aut}(\mathbb{C}^m, \mathbf{w}_1)$ as in (15.1) such that $\tau(Y_1)$ is either Y_2 or its conjugate $\bar{Y}_2$.*

4. Kähler-Einstein metrics on orbifolds

23 (Continuity method for finding Kähler-Einstein metrics)**.** Let (X, Δ) be a compact orbifold of dimension n such that $K_{X^{orb}}^{-1}$ is ample. The continuity method for finding a Kähler-Einstein metric on (X, Δ) was developed by [Aub82, Siu88, Siu87, Tia87, Nad90, DK01].

We start with an arbitrary smooth Hermitian metric h_0 on $K_{X^{orb}}^{-1}$ with positive definite curvature form θ_0. Choose a Kähler metric ω_0 such that $\mathrm{Ricci}(\omega_0) = \theta_0$.

Since θ_0 and ω_0 represent the same cohomology class, there is a C^∞ function f such that

$$\omega_0 = \theta_0 + \tfrac{i}{2\pi}\partial\bar\partial f.$$

Our aim is to find a family of functions ϕ_t and numbers C_t for $t \in [0,1]$, normalized by the condition $\int_X \phi_t \omega_0^n = 0$, such that they satisfy the Monge–Ampère equation

$$\log \frac{(\omega_0 + \frac{i}{2\pi}\partial\bar\partial\phi_t)^n}{\omega_0^n} + t(\phi_t + f) + C_t = 0.$$

We start with $\phi_0 = 0, C_0 = 0$ and if we can reach $t = 1$, we get a Kähler-Einstein metric

$$\omega_1 = \omega_0 + \tfrac{i}{2\pi}\partial\bar\partial\phi_1.$$

It is easy to see that solvability is an open condition on $t \in [0,1]$, the hard part is closedness. It turns out that the critical step is a 0th order estimate. That is, as the values of t for which the Monge–Ampère equation is solvable approach a critical value $t_0 \in [0,1]$, a subsequence of the ϕ_t converges to a function ϕ_{t_0} which is the sum of a C^∞ and of a plurisubharmonic function. We only need to prove that

$$\int_X e^{-\gamma\phi_{t_0}}\omega_0^n < +\infty \quad \text{for some } \gamma > \tfrac{n}{n+1}. \tag{23.1}$$

We view $h_0 e^{-\phi_{t_0}}$ as a singular metric on $K^{-1}_{X^{orb}}$. Its curvature current $\theta_0 + \frac{i}{2\pi}\partial\bar\partial\phi_{t_0}$ is easily seen to be semi positive.

The method is thus guaranteed to work if there is no singular metric with semi positive curvature on $K^{-1}_{X^{orb}}$ for which the integral in (23.1) is divergent.

A theorem of Demailly and Kollár establishes how to approximate a plurisubharmonic function by sums of logarithms of absolute values of holomorphic functions [DK01]. This allows us to replace an arbitrary plurisubharmonic function ϕ_{t_0} by $\frac{1}{s}\log|\tau_s|$, where τ_s is holomorphic. This gives the following criterion:

Theorem 24. [DK01] *Let X^{orb} be a compact, n-dimensional orbifold such that $K^{-1}_{X^{orb}}$ is ample. The continuity method produces a Kähler-Einstein metric on X^{orb} if the following holds:*

There is a $\gamma > \frac{n}{n+1}$ such that for every $s \geq 1$ and for every holomorphic section $\tau_s \in H^0(X^{orb}, K^{-s}_{X^{orb}})$ the following integral is finite:

$$\int |\tau_s|^{-\frac{2\gamma}{s}}\omega_0^n \ < \ +\infty.$$

For the hypersurface case considered in §12 we can combine this with the description of sections of $H^0(X^{orb}, K^{-s}_{X^{orb}})$ given in Proposition 13 to make the condition even more explicit:

Corollary 25. *Let $Y = (F(z_1,\dots,z_m) = 0)$ be as in §12. Assume that $w(F) < \sum w_i$. The continuity method produces a Kähler-Einstein metric on $Y/\mathbb{C}^*$ if the following holds:*

There is a $\gamma > \frac{n}{n+1}$ such that for every weighted homogeneous polynomial g of weighted degree $s(\sum w_i - w(F))$, not identically zero on Y, the function

$$|g|^{-\gamma/s} \quad \text{is locally } L^2 \text{ on } Y \setminus \{0\}.$$

In general it is not easy to decide if a given function $|g|^{-c}$ is locally L^2 or not, but we at least have the following easy criterion. (See, for instance, [Kol97, 3.14, 3.20].)

Lemma 26. *Let M be a complex manifold and h a holomorphic function on M. If $c \cdot \operatorname{mult}_p h < 1$ for every $p \in M$ then $|h|^{-c}$ is locally L^2.*

For g as in Corollary 25 it is relatively esay to estimate the multiplicities of its zeros via intersection theory, and we obtain the following generalization of [JK01, Prop.11].

Proposition 27. *Let $Y = (F(z_1, \dots, z_m) = 0)$ be as in §12. Assume that the intersections of Y with any number of hyperplanes $(z_i = 0)$ are all smooth outside the origin. Let g be a weighted homogeneous polynomial and pick $\delta_i > 0$. Then*

$$|g|^{-c} \prod_i |z_i|^{\delta_i - 1} \quad \text{is locally } L^2 \text{ on } Y \setminus \{0\}$$

if $c \cdot w(F) \cdot w(g) < \min_{i,j}\{w_i w_j\}$.

Proof. The case when every $\delta_i = 1$ is [JK01, Prop.11] combined with Lemma 26. These also show that in our case the L^2-condition holds away from the hyperplanes $(z_i = 0)$.

We still need to check the L^2 condition along the divisors $H_i := (z_i = 0) \cap Y \setminus \{0\}$. This is accomplished by reducing the problem to an analogous problem on H_i and using induction.

In algebraic geometry, this method is called *inversion of adjunction*. Conjectured by Shokurov, the following version is due to Kollár [Kol92, 17.6]. It was observed by [Man93] that it can also be derived from the L^2-extension theorem of Ohsawa and Takegoshi [OT87]. See [Kol97] or [KM98] for more detailed expositions.

Theorem 28 (Inversion of adjunction)**.** *Let M be a smooth manifold, $H \subset M$ a smooth divisor with equation $(h = 0)$ and g a holomorphic function on M. Let g_H denote the restriction of g to H and assume that it is not identically zero. The following are equivalent:*

(1) *$|g|^{-c}|h|^{\delta - 1}$ is locally L^2 near H for every $\delta > 0$.*
(2) *$|g_H|^{-c}$ is locally L^2 on H.*

□

5. Differential Topology of Links

In this section we briefly describe the differential topology of odd dimensional spheres that can be realized as links of Brieskorn–Pham singularities and discuss methods for determining their diffeomorphism type.

29 (The group bP_{2n})**.** The essential work here is that of Kervaire and Milnor [KM63] who showed that associated with each sphere S^n with $n \geq 5$ there is an Abelian group Θ_n consisting of equivalence classes of homotopy spheres S^n that are equivalent under oriented h-cobordism. By Smale's h-cobordism theorem this implies equivalence under oriented diffeomorphism. The group operation on Θ_n is connected sum. Θ_n has a subgroup bP_{n+1} consisting of equivalence classes of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor [KM63] proved that $bP_{2k+1} = 0$ for $k \geq 1$. Moreover, for $m \geq 2$, bP_{4m} is cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator}\left(\frac{4B_m}{m}\right),$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$. In the first two cases these include all exotic spheres; whereas, in the last case $|bP_{16}|$ is precisely half of the homotopy spheres.

For bP_{4m+2} the situation is still not entirely understood. It entails computing the Kervaire invariant, which is hard. It is known (see the recent review paper [Lan00] and references therein) that $bP_{4m+2} = 0$ or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $m \neq 2^i - 1$ for any $i \geq 3$. Furthermore, bP_{4m+2} vanishes for $m = 1, 3, 7$, and 15.

To a sequence $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}_+^m$ Brieskorn associates a graph $G(\mathbf{a})$ whose m vertices are labeled by $a_1, \dots, a_m$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i, a_j) > 1$. Let $G(\mathbf{a})_{ev}$ denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to $G(\mathbf{a})_{ev}$, but $G(\mathbf{a})_{ev}$ may contain odd vertices as well.

Theorem 30. [Bri66] *The link $L(\mathbf{a})$ (with $m \geq 3$) is homeomorphic to the $(2m-3)$-sphere if and only if either of the following hold.*

(1) *$G(\mathbf{a})$ contains at least two isolated points, or*
(2) *$G(\mathbf{a})$ contains a unique odd isolated point and $G(\mathbf{a})_{ev}$ has an odd number of vertices with $\gcd(a_i, a_j) = 2$ for any distinct $a_i, a_j \in G(\mathbf{a})_{ev}$.*

31 (Diffeomorphism types of the links $L(\mathbf{a})$)**.** In order to distinguish the diffeomorphism types of the links $L(\mathbf{a})$ we need to treat the cases $m = 2k+1$ and $m = 2k$ separately.

By [KM63], the diffeomorphism type of a homotopy sphere Σ in bP_{4k} is determined by the signature (modulo $8|bP_{4k}|$) of a parallelizable manifold M whose boundary is Σ. By the Milnor Fibration Theorem [Mil68], if $\Sigma = L(\mathbf{a})$, we can take M to be the Milnor fiber $M^{4k}(\mathbf{a})$ which for links of isolated singularities coming from weighted homogeneous polynomials is diffeomorphic to the hypersurface $\{\mathbf{z} \in \mathbb{C}^m \mid f(z_1, \cdots, z_m) = 1\}$.

Brieskorn shows that the signature of $M^{4k}(\mathbf{a})$ can be written combinatorially as

$$\begin{aligned} \tau(M^{4k}(\mathbf{a})) &= \#\Big\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 0 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 1 \mod 2\Big\} \\ &- \#\Big\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 1 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 2 \mod 2\Big\}. \end{aligned}$$

Using a formula of Eisenstein, Zagier (cf. [Hir71]) has rewritten this as:

$$\tau(M^{4k}(\mathbf{a})) = \frac{(-1)^k}{N} \sum_{j=0}^{N-1} \cot \frac{\pi(2j+1)}{2N} \cot \frac{\pi(2j+1)}{2a_1} \cdots \cot \frac{\pi(2j+1)}{2a_{2k+1}}, \qquad 31.1$$

where N is any common multiple of the a_i's. It is formula 31.1 that our computer program uses to determine the diffeomorphism type.

For the case of bP_{4k-2} the diffeomorphism type is determined by the so-called *Arf invariant* $C(M^{4k-2}(\mathbf{a})) \in \{0, 1\}$. Brieskorn then proves the following:

Proposition 32. *$C(M^{4k-2}(\mathbf{a})) = 1$ holds if and only if condition 2 of Theorem 30 holds and the one isolated point, say a_0, satisfies $a_0 \equiv \pm 3 \mod 8$.*

Following conventional terminology we say that $L(\mathbf{a})$ is a *Kervaire sphere* if $C(M^{4k-2}(\mathbf{a})) = 1$. A Kervaire sphere is not always exotic, but it is exotic when $bP_{4k-2} = \mathbb{Z}_2$.

6. Brieskorn–Pham singularities

Notation 33. Consider a Brieskorn–Pham singularity $Y(\mathbf{a}) := (\sum_{i=1}^m z_i^{a_i} = 0) \subset \mathbb{C}^m$. Set $C = \operatorname{lcm}(a_i : i = 1, \dots, m)$. $Y(\mathbf{a})$ is invariant under the $\mathbb{C}^*$-action

$$(z_1, \dots, z_m) \mapsto (\lambda^{C/a_1} z_1, \dots, \lambda^{C/a_m} z_m). \tag{33.1}$$

In the notation of Definition 7 we have $w_i = C/a_i$ and $w = w(F) = C$. Thus $Y(\mathbf{a})/\mathbb{C}^*$ is a Fano orbifold iff $1 < \sum_{i=1}^m \frac{1}{a_i}$.

More generally, we consider weighted homogeneous perturbations

$$Y(\mathbf{a}, p) := (\sum_{i=1}^m z_i^{a_i} + p(z_1, \dots, z_m) = 0) \subset \mathbb{C}^m, \quad \text{where } w(p) = C.$$

The genericity condition we need, which is always satisfied by $p \equiv 0$ is:

(GC) The intersections of $Y(\mathbf{a}, p)$ with any number of hyperplanes $(z_i = 0)$ are all smooth outside the origin.

In order to formulate the statement, we further set

$$C^j = \operatorname{lcm}(a_i : i \neq j), \quad b_j = \gcd(a_j, C^j) \quad \text{and} \quad d_j = a_j/b_j.$$

Theorem 34. *The orbifold $Y(\mathbf{a}, p)/\mathbb{C}^*$ is Fano and has a Kähler-Einstein metric if it satisfies condition (GC) and*

$$1 < \sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_{i,j} \Big\{ \frac{1}{a_i}, \frac{1}{b_i b_j} \Big\}.$$

Note that if the a_i are pairwise relatively prime then all the b_i's are 1 and we get the simpler bounds $1 < \sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_i \{\frac{1}{a_i}\}$.

Proof. By Corollary 25 we need to show that for every $s > 0$ and for every weighted homogeneous polynomial g of weighted degree $s(\sum w_i - w(F)) = sC(\sum a_i^{-1} - 1)$, the function

$$|g|^{-\gamma/s} \quad \text{is locally } L^2 \text{ on } Y \setminus \{0\}.$$

Our aim is to reduce this to a problem on a perturbation of the simpler Brieskorn–Pham singularity $Y(\mathbf{b})$.

Lemma 35. *Let g be a weighted homogeneous polynomial with respect to the $\mathbb{C}^*$-action (33.1). Then there is a polynomial G such that*

$$g(z_1, \dots, z_m) = \prod z_i^{e_i} \cdot G(z_1^{d_1}, \dots, z_m^{d_m}).$$

Proof. Note that $C = d_i C^i$. Thus d_i divides $C/a_j = d_i C^i/a_j$ for $j \neq i$ but C/a_i is relatively prime to d_i. Write $g = \prod z_i^{e_i} \cdot g^*$ where g^* is not divisible by any z_i. Thus g^* has a monomial which does not contain z_i, and so its weight is divisible d_i. Thus every time z_i appears, its exponent must be divisible by d_i. □

Applying this to the defining equation of $Y(\mathbf{a}, p)$ we obtain that $p(z_1, \dots, z_m) = p^*(z_1^{d_1}, \dots, z_m^{d_m})$ for some polynomial p^*. Set

$$Y(\mathbf{b}, p^*) := (\sum_{i=1}^m x_i^{b_i} + p^*(x_1, \dots, x_m) = 0) \subset \mathbb{C}^m.$$

We have a map $\pi : Y(\mathbf{a}, p) \to Y(\mathbf{b}, p^*)$ given by $\pi^* x_i = z_i^{d_i}$ and

$$|g| = \pi^* \prod |x_i|^{e_i/d_i} \cdot |G(x_1, \dots, x_m)|.$$

The Jacobian of π has (d_i-1)-fold zero along $(z_i=0)$. Thus

$$|g|^{-\gamma/s} \quad \text{is locally } L^2 \text{ on } Y\setminus\{0\}$$

iff

$$|G|^{-\gamma/s}\cdot\prod|x_i|^{-\frac{\gamma e_i}{sd_i}+\frac{1}{d_i}-1} \quad \text{is locally } L^2 \text{ on } Y^*\setminus\{0\}.$$

The latter condition is guaranteed by Proposition 27. Indeed, first we need that each x_i has exponent bigger than -1. This is equivalent to $e_i<\gamma^{-1}s$. We know that $e_iC/a_i\le \operatorname{wdeg} g = sC(\sum a_i^{-1}-1)$ and so it is enough to know that $\sum a_i^{-1}-1<\frac{m-1}{m-2}\frac{1}{a_i}$. The latter is one of our assumptions.

Note that $w(\sum x_i^{b_i})=B$, where $B:=\operatorname{lcm}(b_1,\dots,b_m)$ and going from $G(x_1,\dots,x_n)$ to $G(z_1^{d_1},\dots,z_m^{d_m})$ multiplies the weighted degree by C/B. Thus $w(G)\le\frac{B}{C}w(g)=sB(\sum\frac{1}{a_i}-1)$. Therefore the last condition

$$c\cdot w(F)\cdot w(g)<\min\{w_i,w_j\}$$

of Proposition 27 becomes

$$\frac{\gamma}{s}\cdot B\cdot sB\left(\sum\frac{1}{a_i}-1\right)<\min\left\{\frac{B}{b_i},\frac{B}{b_i}\right\}.$$

After dividing by B^2, this becomes our other assumption. □

Note 36. As algebraic varieties, $Y(\mathbf{a})/\mathbb{C}^*$ is the same as $Y(\mathbf{b})/\mathbb{C}^*$. In particular, when the a_i are pairwise relatively prime then all the $b_i=1$ hence, as a variety, $Y(\mathbf{a})/\mathbb{C}^*\cong\mathbb{CP}^{m-2}$. The orbifold structure is given by the divisor

$$\sum_{i=1}^{m-1}\Bigl(1-\frac{1}{a_i}\Bigr)(y_i=0)+\Bigl(1-\frac{1}{a_m}\Bigr)(\sum y_i=0).$$

It would be very interesting to write down the corresponding Kähler-Einstein metric explictly. This form would then hopefully give a Kähler-Einstein metric without the required upper bound in Theorem 34.

For most cases we get orbifolds with finite automorphism group:

Proposition 37. *Assume that $m\ge 4$ and all but one of the a_i is at least 3. Then the automorphism group of $\{\sum_i x_i^{a_i}=0\}/\mathbb{C}^*$ is finite.*

Proof. It is enough to prove that there are no continuous families of isomorphisms of the form

$$\tau_t(x_i)=x_i+\sum_{j\ge1}t^jg_{ij}(x_1,\dots,x_m).$$

By assumption

$$\sum_i\tau_t(x_i)^{a_i}=\sum_i x_i^{a_i}.$$

Let j_0 be the smallest j such that $g_{ij_0}\ne0$ for some i and look at the t^{j_0} term in the Taylor expansion of the left hand side:

$$\sum_i x_i^{a_i-1}g_{ij_0}(x_1,\dots,x_m)=0.$$

Note that $w(g_{ij})=w(x_i)$ so as long as $a_i\ge3$ for all but one i, the terms coming from different values of i do not cancel. Thus every $g_{ij_0}=0$, a contradiction. □

Remark 38. More generally, the automorphism group of any $(F=0)/\mathbb{C}^*$ is finite as long as $w_i < \frac{1}{2}w(F)$ for all but one of the w_is and $(F=0)$ is smooth outside the origin. Indeed, in this case we would get a relation $\sum(\partial F/\partial x_i)\cdot g_{ij_0} = 0$. By assumption, the $\partial F/\partial x_i$ form a regular sequence, and so linear relationships with polynomial coefficients between them are generated by the obvious "Koszul" relations $(\partial F/\partial x_i)\cdot(\partial F/\partial x_j) = (\partial F/\partial x_j)\cdot(\partial F/\partial x_i)$. We get a contradiction by degree considerations.

7. Numerical examples

Our ultimate aim is to obtain a complete enumeration of all sequences that yield a Sasakian-Einstein metric on some homotopy sphere. We can accomplish this in low dimensions via a computer program, to be published elsewhere. Here we content ourselves with obtaining some examples which show the double exponential growth of the number of cases.

Example 39. Consider sequences of the form $\mathbf{a} = (2,3,7,m)$. By explicit calculation, the corresponding link $L(\mathbf{a})$ gives a Sasakian-Einstein metric on S^5 if $5 \le m \le 41$ and m is relatively prime to at least two of $2,3,7$. This is satisfied in 27 cases.

Example 40. Among the above cases, the sequence $\mathbf{a} = (2,3,7,35)$ is especially noteworthy. If $C(u,v)$ is any sufficiently general homogenous septic polynomial, then the link of

$$x_1^2 + x_2^3 + C(x_3, x_4^5)$$

also gives a Sasakian-Einstein metric on S^5. The relevant automorphism group of $\mathbb{C}^4$ is

$$(x_1,x_2,x_3,x_4) \mapsto (x_1, x_2, \alpha_3 x_3 + \beta x_4^5, \alpha_4 x_4).$$

Hence we get a $2(8-3) = 10$ real dimensional family of Sasakian-Einstein metrics on S^5.

Similarly, the sequence $\mathbf{a} = (2,3,7,43,43\cdot 31)$ gives a standard 7-sphere with a $2(43-2) = 82$-dimensional family of Sasakian-Einstein metrics on S^7.

Example 41 (Euclid's or Sylvester's sequence)**.** (See [GKP89, Sec.4.3] or [Slo03, A000058].)

Consider the sequence defined by the recursion relation

$$c_{k+1} = c_1 \cdots c_k + 1 = c_k^2 - c_k + 1$$

beginning with $c_1 = 2$. It starts as

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$$

It is easy to see (cf. [GKP89, 4.17]) that

$$c_k \ge (1.264)^{2^k} - \tfrac{1}{2} \quad \text{and} \quad \sum_{i=1}^{m} \frac{1}{c_i} = 1 - \frac{1}{c_{m+1}-1} = 1 - \frac{1}{c_1\cdots c_m}.$$

Example 42. Consider sequences of the form $\mathbf{a} = (a_1 = c_1, \ldots, a_{m-1} = c_{m-1}, a_m)$. The troublesome part of the condition of Theorem 34 is the computation of the b_i. However $b_i \le a_i$ thus it is sufficient to satisfy the following stronger restriction:

$$1 < \sum_{i=1}^{m} \frac{1}{a_i} < 1 + \frac{m-1}{m-2}\min_{i,j}\Big\{\frac{1}{a_i a_j}\Big\} = 1 + \frac{m-1}{m-2}\cdot\frac{1}{a_{m-1}a_m}.$$

By direct computation this is satisfied if $c_m - c_{m-1} < a_m < c_m$. At least a third of these numbers are relatively prime to $a_1 = 2$ and to $a_2 = 3$, thus we conclude:

Proposition 43. *Our methods yield at least* $\frac{1}{3}(c_{m-1}-1) \geq \frac{1}{3}(1.264)^{2^{m-1}} - 0.5$ *inequivalent families of Sasakian-Einstein metrics on (standard and exotic)* $(2m-3)$*-spheres.*

If $2m-3 \equiv 1 \mod 4$ then by Proposition 32, all these metrics are on the standard sphere. If $2m-3 \equiv 3 \mod 4$ then all these metrics are on on both standard and exotic spheres but we cannot say anything in general about their distribution.

Example 44. We consider the sequences $\mathbf{a} = (a_1 = c_1, \dots, a_{m-1} = c_{m-1}, a_m = (c_{m-1}-2)c_{m-1})$. Any two of them are relatively prime, except for $\gcd(a_{m-1}, a_m) = c_{m-1}$, and the conditions of Theorem 34 are satisfied.

The Brieskorn–Pham polynomial has weighted homogeneous perturbations

$$x_1^{a_1} + \cdots + x_{m-2}^{a_{m-2}} + G(x_{m-1}, x_m^{c_{m-1}-2})$$

where G is any homogeneous polynomial of degree c_{m-1}. Up to coordinate changes, these form a family of complex dimension $c_{m-1}-2$. Thus we conclude:

Proposition 45. *Our methods yield an at least* $2(c_{m-1}-2) \geq 2((1.264)^{2^{m-1}} - 2.5)$*-dimensional (real) family of pairwise inequivalent Sasakian-Einstein metrics on some (standard or exotic)* $(2m-3)$*-sphere.*

As before, if $2m-3 \equiv 1 \mod 4$ then these metrics are on the standard sphere.

Example 46. Consider sequences of the form $\mathbf{a} = (a_1 = 2c_1, \dots, a_{m-2} = 2c_{m-2}, a_{m-1} = 2, a_m)$ where a_m is relatively prime to all the other a_is. By easy computation, the condition of Theorem 34 is satisfied if $2c_{m-2} < a_m < 2c_{m-1} - 2$.

The relatively prime condition is harder to pin down, but it certainly holds if in addition a_m is a prime number. By the prime number theorem, the number of primes in the interval $[c_{m-1}, 2c_{m-1}]$ is about

$$\frac{c_{m-1}}{\log c_{m-1}} \geq \frac{(1.264)^{2^{m-2}}}{2^{m-1}\log 1.264} \geq (1.264)^{2^{m-1}-4(m-1)},$$

so it is still doubly exponential in m.

By Proposition 32, for even m, $L(\mathbf{a})$ the standard sphere if $a \equiv \pm 1 \mod 8$ and the Kervaire sphere if $a \equiv \pm 3 \mod 8$. It is easy to check for all values of m that we get at least one solution of both types. Thus we conclude:

Proposition 47. *Our methods yield a doubly exponential number of inequivalent families of Sasakian-Einstein metrics on both the standard and the Kervaire* $(4m-3)$*-spheres.*

Acknowledgments . We thank Y.-T. Siu for answering several questions, and E. Thomas for helping us with the computer programs. CPB and KG were partially supported by the NSF under grant number DMS-0203219 and J.K. was partially supported by the NSF under grant number DMS-0200883. The authors would also like to thank Università di Roma "La Sapienza" for partial support where discussions on this work initiated.

References

[Aub82] Thierry Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 252, Springer-Verlag, New York, 1982.

[Bes87] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.

[BG97] Charles P. Boyer and Krzysztof Galicki, *The twistor space of a 3-Sasakian manifold*, Internat. J. Math. **8** (1997), no. 1, 31–60.

[BG99] Charles Boyer and Krzysztof Galicki, *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, pp. 123–184.

[BG00] Charles P. Boyer and Krzysztof Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909.

[BG01] Charles P. Boyer and Krzysztof Galicki, *New Einstein metrics in dimension five*, J. Differential Geom. **57** (2001), no. 3, 443–463.

[BGN03a] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye, *On the geometry of Sasakian-Einstein 5-manifolds*, Math. Ann. **325** (2003), no. 3, 485–524.

[BGN03b] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye, *Sasakian geometry, homotopy spheres and positive Ricci curvature*, Topology **42** (2003), no. 5, 981–1002.

[BK78] Jean-Pierre Bourguignon and Hermann Karcher, *Curvature operators: pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 1, 71–92.

[Böh98] Christoph Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math. **134** (1998), no. 1, 145–176.

[Bri66] Egbert Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14.

[BW58] W. M. Boothby and H. C. Wang, *On contact manifolds*, Ann. of Math. (2) **68** (1958), 721–734.

[DK01] Jean-Pierre Demailly and János Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556.

[FK90] Thomas Friedrich and Ines Kath, *7-dimensional compact Riemannian manifolds with Killing spinors*, Comm. Math. Phys. **133** (1990), no. 3, 543–561.

[GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics*, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989, A foundation for computer science.

[GM74] Detlef Gromoll and Wolfgang Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. (2) **100** (1974), 401–406.

[Gro68] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), North-Holland Publishing Co., Amsterdam, 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2.

[GZ00] Karsten Grove and Wolfgang Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** (2000), no. 1, 331–367.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Hir71] F. Hirzebruch, *Pontrjagin classes of rational homology manifolds and the signature of some affine hypersurfaces*, Proceedings of Liverpool Singularities Symposium, II (1969/1970) (Berlin), Springer, 1971, (reprinted in F. Hirzebruch, Gesammelte Abhandlungen, band II), pp. 207–212. Lecture Notes in Math., Vol. 209.

[Jen73] Gary R. Jensen, *Einstein metrics on principal fibre bundles*, J. Differential Geometry **8** (1973), 599–614.

[JK01] J. M. Johnson and J. Kollár, *Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 69–79.

[KM63] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537.

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kob63] Shoshichi Kobayashi, *Topology of positively pinched Kaehler manifolds*, Tôhoku Math. J. (2) **15** (1963), 121–139.

[Kol92] János Kollár, *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Kol97] János Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.

[KS88] Matthias Kreck and Stephan Stolz, *A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with* $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$*-symmetry*, Ann. of Math. (2) **127** (1988), no. 2, 373–388.

[Lan00] Timothy Lance, *Differentiable structures on manifolds*, Surveys on surgery theory, Vol. 1, Ann. of Math. Stud., vol. 145, Princeton Univ. Press, Princeton, NJ, 2000, pp. 73–104.

[Man93] Laurent Manivel, *Un théorème de prolongement* L^2 *de sections holomorphes d'un fibré hermitien*, Math. Z. **212** (1993), no. 1, 107–122.

[Mil56] John Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405.

[Mil68] John Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968.

[Nad90] Alan Michael Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596.

[OT87] Takeo Ohsawa and Kenshō Takegoshi, *On the extension of* L^2 *holomorphic functions*, Math. Z. **195** (1987), no. 2, 197–204.

[Siu87] Yum Tong Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, DMV Seminar, vol. 8, Birkhäuser Verlag, Basel, 1987.

[Siu88] Yum Tong Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) **127** (1988), no. 3, 585–627.

[Slo03] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, AT&T Research, http://www.research.att.com/ njas/sequences/, 2003.

[Tak78] Takao Takahashi, *Deformations of Sasakian structures and its application to the Brieskorn manifolds*, Tôhoku Math. J. (2) **30** (1978), no. 1, 37–43.

[Tia87] Gang Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $C_1(M) > 0$, Invent. Math. **89** (1987), no. 2, 225–246.

[Wra97] David Wraith, *Exotic spheres with positive Ricci curvature*, J. Differential Geom. **45** (1997), no. 3, 638–649.

[Zil82] W. Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), no. 3, 351–358.

CPB and KG: Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.

E-mail address: cboyer@math.unm.edu

E-mail address: galicki@math.unm.edu

JK: Department of Mathematics, Princeton University, Princeton, NJ 08544-1000.

E-mail address: kollar@math.princeton.edu